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Primitive Ideals of the Coordinate Ring of Quantum Symplectic Space

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The primitive ideals of the coordinate ring of quantum symplectic space $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ are classified when q is not a root of unity. It is shown that all primitive ideals of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ correspond to its admissible sets, the center of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ is just $\mathbf{C}$, and the Gelfand–Kirillov dimensions of primitive factored algebras of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ are even.

0. INTRODUCTION

M. Takeuchi constructed the quantum orthogonal group and the quantum symplectic group in [T] by using the matrices which are symmetric satisfying the braid condition as well as having three different eigenvalues. He also defined some q-analogues $S_q^+(V), S_q^-(V)$ of the symmetric algebras as I. Yu. Manin did to reformulate the quantum algebras $\mathcal{O}_q[M_n(\mathbf{C})]$, where V is an n-dimensional vector space over the complex numbers $\mathbf{C}$. Similarly, L. D. Faddeev, *et al.* found the same algebras in [FRT]. S. P. Smith called the algebra $S_q^+(V)$ the coordinate ring of quantum Euclidean space, and the algebra $S_q^-(V)$ the coordinate ring of quantum symplectic space, denoted by $\mathcal{O}_q(\mathfrak{o}\mathbf{C}^n)$ and $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^n)$, respectively.

In this paper, we classify all the primitive ideals of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ by using *admissible sets* when q is not a root of unity (see 1.4 for the definition of admissible sets). This is the answer to S. P. Smith's question about the primitive ideals of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ which is posed in the last section of [S1]. For classifying all primitive ideals, first we find candidates $P_T(\boldsymbol{\alpha})$ for primitive ideals, and then prove their primeness, and finally, we see that, from [MR, 9.1.8] or [S2, 8.8], $P_T(\boldsymbol{\alpha})$ is primitive by showing that the intersection of every prime ideal strictly containing $P_T(\boldsymbol{\alpha})$ strictly contains $P_T(\boldsymbol{\alpha})$.

In 1.10, we observe that $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ is an iterated Ore extension over the complex numbers $\mathbf{C}$. Thus the basic tools to be used are theories developed in skew polynomial rings, for instance, left Ore sets, prime factors, Gelfand–Kirillov dimension, etc. A good reference for the relevant

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basic ideas, including skew polynomial rings, left Ore sets, and Gelfand–Kirillov dimension, is [MR]. In the Appendix, we outline the admissible sets and corresponding primitive ideals of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^4)$. These examples will help readers understand the ideas of the proofs.

We assume throughout that *q is not a root of unity*, ground fields of all considered algebras are the complex numbers $\mathbf{C}$, and all algebras have unity.

1. Definitions and Basic Properties

In this section, we give a definition of the coordinate ring of quantum symplectic space $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$, defined in [FRT, T], find certain elements $X_1, \dots, X_{2n}, \Omega_1, \dots, \Omega_n$ which work basically for classifying the primitive ideals in 1.3, and observe easily in 1.10 that $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ is an iterated Ore extension over the complex numbers $\mathbf{C}$. We also give several definitions and notations which are useful in 1.4 and 1.6.

Definition 1.1. (See [FRT, T]). The coordinate ring of quantum symplectic space $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ is the algebra generated by $X_1, \dots, X_{2n}$ satisfying the following relations: Set $i' = 2n + 1 - i$ for $1 \le i \le 2n$. Then the defining relations are

$$X_j X_i = q X_i X_j, \qquad i < j, i' \neq j,$$
$$X_{i'} X_i = q^2 X_i X_{i'} + (q^2 - 1) \sum_{i \le k < i} q^{i-k} X_k X_{k'}, \qquad i < i'.$$

Definition 1.2. Let R be an algebra and $y \in R$. We will say that y is *normal* if $Ry = yR$. Let Z be a subset of R. We will say that y is *Z-normal* if y is normal and there is a nonzero complex number α such that $zy = \alpha yz$ for each $z \in Z$. If Z is a singleton $\{z\}$, then y will be said to be just z-normal.

Lemma 1.3. *Put $\Omega_i = \Sigma_{1 \le k \le i} q^{i-k} X_k X_{k'}$, $i = 1, \dots, n$.*

(1) *For a fixed Ω_i, $1 \le i \le n$,*

$$\Omega_i X_k = q^2 X_k \Omega_i, 1 \le k \le i; \qquad \Omega_i X_k = q^{-2} X_k \Omega_i, i' \le k \le 2n;$$
$$\Omega_i X_k = X_k \Omega_i, i < k < i'; \qquad \Omega_i \Omega_k = \Omega_k \Omega_i, 1 \le k \le n.$$

Hence Ω_i is a $\{X_i | i = 1, 2, 3, \dots, 2n\}$-normal element of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$.

(2) *We have the following rules*:

$$\Omega_i = \sum_{j+1 \le k \le i} q^{i-k} X_k X_{k'} + q^{i-j} \Omega_j,$$

$$X_{i'} X_i - q^2 X_i X_{i'} = (q^2 - 1) q \Omega_{i-1},$$

$$X_{i'} X_i - X_i X_{i'} = (q^2 - 1) \Omega_i.$$

Hence the images of $X_i, X_{i'}, X_{i-1}, X_{(i-1)'}$ *are* $\{\overline{X}_j | j = 1, 2, 3, \ldots, 2n\}$*-normal elements of algebra* $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle \Omega_{i-1} \rangle$.

Proof. We have the above formulas from the basic definition of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ by calculation. ∎

DEFINITION 1.4. Let $\wp = \{X_1, X_{1'}, \ldots, X_n, X_{n'}, \Omega_1, \ldots, \Omega_n\}$ be a subset of the algebra $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$. We will say that a subset $T \subseteq \wp$ is *admissible* if T satisfies the following two conditions:

(1) $X_i \in T$ or $X_{i'} \in T$ if and only if $\Omega_i \in T$ and $\Omega_{i-1} \in T, 2 \le i \le n$.

(2) $X_1 \in T$ or $X_{1'} \in T$ if and only if $\Omega_1 \in T$.

For instance, $\{X_3, \Omega_2, \Omega_3\}, \{X_6, \Omega_1, \Omega_3\}$ are admissible sets in $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^6)$, in which we have $1' = 6$, $2' = 5$, $3' = 4$.

PROPOSITION 1.5. *Let P be a prime ideal of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$. Then $P \cap \wp$ is an admissible set.*

Proof. This follows from 1.3(2) immediately. ∎

DEFINITION 1.6. Let T be an admissible set.

(1) We denote by $\operatorname{ind}(T)$ the set $\{i | \Omega_i \in T\}$.

(2) We will say that T is *connected* if $\operatorname{ind}(T)$ satisfies the condition: if $i < j < k$ and $i, k \in \operatorname{ind}(T)$ then $j \in \operatorname{ind}(T)$. For instance, $\{X_3, \Omega_2, \Omega_3\}$ is connected in $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^6)$ but $\{X_6, \Omega_1, \Omega_3\}$ is not connected.

(3) Let S be a connected admissible subset of T. We will say that S is *connected component* of T if S satisfies the condition: if U is connected admissible and $S \subseteq U \subseteq T$, then $S = U$. For instance, the connected components of the admissible set $\{X_6, \Omega_1, \Omega_3\}$ in $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^6)$ are $\{X_6, \Omega_1\}$ and $\{\Omega_3\}$.

(4) Let $i \in \operatorname{ind}(T)$. If $X_i, X_{i'} \in T$, then i will be called *removable*.

LEMMA 1.7. *Any admissible set is a disjoint union of its connected components. Such an expression will be called the connected decomposition.*

1.8. Let T have connected decomposition $T_1 \cup \cdots \cup T_r$. Then we always assume that if $i_1 \in \operatorname{ind}(T_i)$, $j_1 \in \operatorname{ind}(T_j)$, and $i < j$ then $i_1 < j_1$.

1.9. When we mention iterated Ore extension ($:=$ iterated skew polynomial ring) $R[Y_1, \sigma_1, \delta_1]\cdots[Y_p, \sigma_p, \delta_p]$, we assume throughout that σ_i are automorphisms and δ_i are left σ_i-derivations. Moreover we write $R[Y_1, \dots, Y_p]$ for $R[Y_1, \sigma_1, \delta_1]\cdots[Y_p, \sigma_p, \delta_p]$ if the automorphisms and derivations are clear. Refer to [GW, Chap. 1; MR, 1.2] for basic properties of a skew polynomial ring and an iterated Ore extension.

PROPOSITION 1.10. (1) *The algebra* $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ *is an iterated Ore extension. Thus it is a noetherian and integral domain.*

(2) *There is a* $\mathbf{C}$*-basis for* $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ *consisting of*

$$\mathscr{A} = \{X_1^{r_1}X_2^{r_2}\cdots X_{2n}^{r_{2n}} \mid r_i = 0, 1, \dots\}.$$

Proof. Observe that $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ is an iterated Ore extension

$$\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n}) = \mathbf{C}[X_1][X_{1'}, \sigma_{1'}, \delta_{1'}]\cdots[X_n, \sigma_n, \delta_n][X_{n'}, \sigma_{n'}, \delta_{n'}]$$

for automorphisms σ_i and σ_i-derivations δ_i defined as follows: For each $1 \le i \le n$,

$$\sigma_i\colon \mathbf{C}[X_1, X_{1'}, \dots, X_{i-1}, X_{(i-1)'}] \to \mathbf{C}[X_1, X_{1'}, \dots, X_{i-1}, X_{(i-1)'}],$$

$$\sigma_i(X_j) = \begin{cases} qX_j, & 1 \le j \le i-1, \\ q^{-1}X_j, & (i-1)' \le j \le 1', \end{cases}$$

$$\delta_i\colon \mathbf{C}[X_1, X_{1'}, \dots, X_{i-1}, X_{(i-1)'}] \to \mathbf{C}[X_1, X_{1'}, \dots, X_{i-1}, X_{(i-1)'}],$$

$$\delta_i(X_j) = 0, \qquad 1 \le j \le i-1, (i-1)' \le j \le 1',$$

$$\sigma_{i'}\colon \mathbf{C}[X_1, X_{1'}, \dots, X_{i-1}, X_{(i-1)'}, X_i] \to \mathbf{C}[X_1, X_{1'}, \dots, X_{i-1}, X_{(i-1)'}, X_i],$$

$$\sigma_{i'}(X_j) = \begin{cases} qX_j, & 1 \le j \le i-1, \\ q^{-1}X_j, & (i-1)' \le j \le 1', \\ q^2X_i, & j = i, \end{cases}$$

$$\delta_{i'}\colon \mathbf{C}[X_1, X_{1'}, \dots, X_{i-1}, X_{(i-1)'}, X_i] \to \mathbf{C}[X_1, X_{1'}, \dots, X_{i-1}, X_{(i-1)'}, X_i],$$

$$\delta_{i'}(X_j) = \begin{cases} 0, & j \ne i, \\ (q^2-1)\Sigma_{1 \le k < i} q^{i-k} X_k X_{k'}, & j = i. \end{cases} \qquad \blacksquare$$

DEFINITION 1.11. Let S be a $\mathbf{C}$-algebra, R a subset of S, and let $Y_1, \dots, Y_m$ be elements of S. Then the algebra S will be called a *PBW-algebra* of type I with variables $Y_1, \dots, Y_m$ over R, $I \subseteq (\mathbb{Z}_+)^m$ if $Y_iR = RY_i$ for all $i = 1, \dots, m$ and every element of S is expressed uniquely as a combination of elements of the set $\{Y_1^{k_1} \cdots Y_m^{k_m} | (k_1, \dots, k_m) \in I\}$ over R.

If $f \in S$ then there exist unique nonzero elements $f_1, \dots, f_n \in R$ and $a_1, \dots, a_n \in \{Y_1^{k_1} \cdots Y_m^{k_m} | (k_1, \dots, k_n) \in I\}$ such that $f = f_1 a_1 + \cdots + f_n a_n$. We shall say that *monomials* of f are $a_1, \dots, a_n$.

COROLLARY 1.12. *The algebra $\mathscr{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ is a PBW-algebra over* $\mathbf{C}$ *of type $(\mathbb{Z}_+)^{2n}$ with variables $X_1, \dots, X_{2n}$ and its Gelfand–Kirillov dimension is* $2n$.

Proof. It is straightforward to calculate the Gelfand–Kirillov dimension of $\mathscr{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ by 1.10(2). (Refer to [MR, Chap. 8] for the definition and properties of the Gelfand–Kirillov dimension.)

THEOREM 1.13. *All prime ideals of $\mathscr{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ are completely prime.*

Proof. It is easy to see that $\sigma_i \delta_i = q^{-2} \delta_i \sigma_i$. Since the subgroup of $\mathbf{C}^*$ generated by all entries of the matrix

$$\begin{pmatrix} 1 & q^{-2} & q^{-1} & q^{-1} & \cdots & q^{-1} & q^{-1} \\ q^2 & 1 & q & q & \cdots & q & q \\ q & q^{-1} & 1 & q^{-2} & \cdots & q^{-1} & q^{-1} \\ q & q^{-1} & q^2 & 1 & \cdots & q & q \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q & q^{-1} & q & q^{-1} & \cdots & 1 & q^{-2} \\ q & q^{-1} & q & q^{-1} & \cdots & q^2 & 1 \end{pmatrix}$$

is torsion free, the result follows from [GL, 2.3]. ∎

2. IDEAL GENERATED BY AN ADMISSIBLE SET

This section serves for proving the fact that the ideal generated by an admissible set is completely prime (see 2.7). The proof uses the Euclidean algorithm and the fact that a skew polynomial ring over a domain is also a domain.

LEMMA 2.1. *Let a noetherian* $\mathbf{C}$*-algebra S be an integral domain, $\sigma \in \operatorname{Aut}(S)$ and let δ be a σ-derivation. Put $U = S[Y, \sigma, \delta]$ the skew polynomial ring over S. If $aY + b \in U$ is a normal element and a is invertible in S, then the ideal $\langle aY + b \rangle$ is completely prime in U.*

Proof. Notice that $\deg(fg) = \deg(f) + \deg(g)$ for nonzero $f, g \in U$. Given $f \in U$, there are $h \in U, r \in S$ such that $f = h(aY + b) + r$ by the

Euclidean algorithm, because a is invertible. Given $f = h_1(aY + b) + r_1$ and $g = h_2(aY + b) + r_2$, if $fg \in \langle aY + b \rangle$ then $r_1 r_2 \in \langle aY + b \rangle$. Thus $r_1 r_2 = 0$. Since S is a domain, $r_1 = 0$ or $r_2 = 0$ and thus $f \in \langle aY + b \rangle$ or $g \in \langle aY + b \rangle$. ∎

LEMMA 2.2. *Let S and U be as in* 2.1. *If $aY + b \in U$ is Y-normal, the set $\mathfrak{I} = \{a^k | k = 0, 1, 2, \dots\}$ is a (left) Ore set in S and $a\delta(a) = ca$ for some $c \in S$. Then $\mathfrak{I} = \{a^k | k = 0, 1, 2, \dots\}$ is a (left) Ore set in U.*

Proof. Since $aY + b \in U$ is Y-normal, we have $Y(aY + b) = \alpha(aY + b)Y$ for some $\alpha \in \mathbf{C}^*$. Thus $\sigma(a) = \alpha a$. Since we have $\alpha a^2 Y = (aY - c)a$ from the definition of skew polynomial ring, the set $\mathfrak{I} = \{a^k | k = 0, 1, 2, \dots\}$ is a (left) Ore set in U. ∎

PROPOSITION 2.3. *Let $S, U, aY + b, a, \mathfrak{I}$ be as in 2.2. Assume that S is a PBW-algebra of type $(\mathbb{Z}_+)^m$ with variables $Y_1, \dots, Y_m$ over a subset R. Hence all elements of S are expressed uniquely as a combination of monomials of Y_i's over R. Let $Y_i Y_j = \lambda_{ij} Y_j Y_i$, $\lambda_{ij} \in \mathbf{C}$, and let $a = \beta Y_{i_1}^{s_{i_p}} \cdots Y_{i_p}^{s_{i_p}}$, $\beta \in \mathbf{C}^*$. If $0 \neq b \in R$, then the ideal $\langle aY + b \rangle$ is completely prime in U.*

Proof. By 2.1, 2.2, the ideal $\langle aY + b \rangle$ is completely prime in the localization $U_{\mathfrak{I}}$. Given $f, g \in U$, suppose that $fg \in \langle aY + b \rangle$. Then $f \in \langle aY + b \rangle U_{\mathfrak{I}}$ or $g \in \langle aY + b \rangle U_{\mathfrak{I}}$, say $f \in \langle aY + b \rangle U_{\mathfrak{I}}$. Thus $a^k f = h(aY + b)$ for some $h \in U$ and k. Put $h = h_0 + h_1 Y + \cdots + h_l Y^l$. We may assume that there is an h_i which is not left divisible by a. Moreover, we have

$$h(aY + b) = h_0(aY + b) + \alpha h_1(aY + b)Y + \cdots + a^l h_l(aY + b)Y^l,$$

and $aR = Ra$ since $Y(aY + b) = \alpha(aY + b)Y$ for some $\alpha \in \mathbf{C}^*$. Write down all coefficients of $h(aY + b)$ as R-combinations of monomials expressed by Y_i's and Y. Then k must be zero. That is, $f \in \langle aY + b \rangle$. ∎

2.4. Hereafter, we will use the same notation in $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ and its factor algebras if no confusion is likely to arise. For instance, if I is an ideal of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$, then $X_1 \in \mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/I$ means the image of X_1 by the canonical map $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n}) \to \mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/I$.

LEMMA 2.5. *Let $k < n$ and let U be the subalgebra of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ generated by $X_1, X_{1'}, \dots, X_k, X_{k'}$. If I is an ideal of U such that $\sigma_i(I) \subseteq I$ and $\delta_i(I) \subseteq I$ for all $k + 1 \leq i \leq (k + 1)'$, where σ_i and δ_i are as defined in the proof of* 1.10, *then $I\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n}) = \mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})I$ and $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/I\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ is isomorphic to the iterated Ore extension $(U/I)[X_{k+1}, X_{(k+1)'}, \dots, X_n, X_{n'}]$.*

Proof. Since the ideal I is σ_i, δ_i-stable, there are automorphisms σ_i' and σ_i'-derivations δ_i' of suitable factor algebras. ∎

LEMMA 2.6. *The ideal* $\langle \Omega_i \rangle$ *of* $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$, $i > 1$, *is completely prime.* –

Proof. By 2.5, it is enough to show that $\langle \Omega_i \rangle$ is a completely prime ideal of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2i})$. Thus we may assume that $i = n > 1$.

In this case the result follows from 2.3 with $R = \mathbf{C}[X_1, X_{1'}, \dots X_{n-1}, X_{(n-1)'}]$, $S = R[X_n]$, $U = R[X_n][X_{n'}] = \mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$, $Y = X_{n'}$, and $aY + b = \Omega_n = X_n X_{n'} + q\Omega_{n-1}$. ∎

THEOREM 2.7. *Let T be an admissible set of* $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$. *Then the ideal* $\langle T \rangle$ *is completely prime.*

Proof. Let $T = T_1 \cup \cdots \cup T_r$ be the connected decomposition. We use induction on r. If $r = 0$, there is nothing to do. Assume that $r > 0$ and that $I = \langle T_1 \cup \cdots \cup T_k \rangle$ is completely prime in the subalgebra U of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ generated by $X_1, X_{1'}, \dots, X_i, X_{i'}$, where $i = \max(\operatorname{ind}(T_k))$. Set $j = \min(\operatorname{ind}(T_{k+1}))$, $l = \max(\operatorname{ind}(T_{k+1}))$. If $j = 1$, then observe that $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T_1 \cup \cdots \cup T_{k+1} \rangle$ is an iterated Ore extension. Suppose that $j > 1$. Then $\langle \Omega_j \rangle$ is completely prime in the algebra $(U/I)[X_{i+1}, X_{(i+1)'}, \dots, X_j, X_{j'}]$ as in the proof of 2.6. Thus $J = \langle I, \Omega_j \rangle$ is completely prime in $U[X_{i+1}, X_{(i+1)'}, \dots, X_j, X_{j'}] = S$. From 2.5 and the proof of 1.10, the factor algebra $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T_1 \cup \cdots \cup T_{k+1} \rangle$ is isomorphic to the algebra$(S/J)[X_{i_1}, \dots, X_{i_t}][X_{l+1}, X_{(l+1)'}, \dots, X_n, X_{n'}]$ where the set $\{X_{i_1}, \dots, X_{i_t}\} = \{X_{j+1}, X_{(j+1)'}, \dots, X_l, X_{l'}\} - T_{k+1}$, $i_s < i_{s+1}$. Hence the ideal $\langle T_1 \cup \cdots \cup T_{k+1} \rangle$ is completely prime. ∎ –

3. GELFAND–KIRILLOV DIMENSION OF $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T \rangle$

In this section, we analyze the Gelfand–Kirillov dimension of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T \rangle$ for every admissible set T. In 3.3, we see that the Gelfand–Kirillov dimension of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T \rangle$ depends on only length(T) which we can find easily. (See 3.1 for the definition of length.)

DEFINITION 3.1. Let T be an admissible set of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$. Define

(1) $\operatorname{ht}(T) =$ maximal length of a chain of admissible subsets of T.

(2) $\operatorname{depth}(T) =$ maximal length of a chain of admissible sets containing T.

(3) Let T be connected and $i = \min(\operatorname{ind}(T))$. Set $S = T \cap \{X_1, X_{1'}, \dots, X_n, X_{n'}\}$.

We define length(T) to be

$$\text{length}(T) = \begin{cases} |S| & \text{if } i = 1 \\ |S| + 1 & \text{if } i \neq 1. \end{cases}$$

Let $T_1 \cup \cdots \cup T_r$ be the connected decomposition of T. We define length(T) to be length(T) $= \Sigma_{1 \le k \le r}$length(T_k).

(4) For any algebra R, the Gelfand–Kirillov dimension of R is denoted by $GK(R)$.

PROPOSITION 3.2. *For any admissible set* T, $\mathrm{ht}(T) + \mathrm{depth}(T) = 2n$.

Proof. By 1.12, 2.7, and [MR, 8.3.6(i)], it is enough to show that there is a chain of admissible sets of length $2n$ such that some component is T. We prove this by induction on n. Set $\wp_i = \{X_1, X_{1'}, \ldots, X_i, X_{i'}, \Omega_1, \ldots, \Omega_i\}$, $1 \le i \le n$, and $\tilde{T} = T \cap \wp_{n-1}$. Notice that the subalgebra generated by $\wp_{n-1}$ is isomorphic to $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n-2})$ and $\tilde{T}$ is an admissible set of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n-2})$.

Case 1. $\Omega_{n-1} \notin T$: By induction there is a chain $\tilde{T}_i$ of admissible sets of length $2n-2$ such that some component is $\tilde{T}$. If $\Omega_n \notin T$, then we have $T = \tilde{T}$ and the following chain of length $2n$

$$\varnothing \subset \cdots \subset \tilde{T}_{2n-2} = \wp_{n-1} \subset (\wp_{n-1} \cup \{X_n, \Omega_n\}) \subset \wp_n.$$

If $\Omega_n \in T$, we have $T = \tilde{T} \cup \{\Omega_n\}$. Set

$$T_i' = \begin{cases} \tilde{T}_i \cup \{\Omega_n\} & \text{if } \Omega_{n-1} \notin \tilde{T}_i, \\ \tilde{T}_i \cup \{X_n, \Omega_n\} & \text{if } \Omega_{n-1} \in \tilde{T}_i. \end{cases}$$

Then T_i' is an admissible set and we have the following chain of length $2n$

$$\varnothing \subset \{\Omega_n\} \subset T_1' \subset \cdots \subset T_{2n-2}' \subset \wp_n.$$

Case 2. $\Omega_{n-1} \in T$: There is a chain $\tilde{T}_i$ of length $2n-2$ such that $\tilde{T}_i = \tilde{T}$ for some i by induction hypothesis. If $\Omega_n \notin T$, then we have $T = \tilde{T}$ and the following chain of length $2n$

$$\varnothing \subset \cdots \subset \tilde{T}_{2n-2} = \wp_{n-1} \subset (\wp_{n-1} \cup \{X_n, \Omega_n\}) \subset \wp_n.$$

If $\Omega_n \in T$, we have $T = \tilde{T} \cup T'$ where

$$T' = \begin{cases} \{X_n, \Omega_n\} \\ \{X_{n'}, \Omega_n\} \\ \{X_n, X_{n'}, \Omega_n\}. \end{cases}$$

Let k be the first index such that $\Omega_{n-1} \in \tilde{T}_i$. Suppose that $T' = \{X_n, \Omega_n\}$. Then $\tilde{T}_i \cup T'$ is also an admissible set for each $i \geq k$ and so we have the following chain of length $2n$

$$\varnothing \subset \cdots \subset \tilde{T}_{k-1} \subset \tilde{T}_k \subset \left(\tilde{T}_k \cup T'\right) \cup \cdots \cup \left(\tilde{T}_{2n-2} \cup T'\right) \subset \wp_n.$$

The case $T' = \{X_{n'}, \Omega_n\}$ is similar to the above one. Suppose that $T' = \{X_n, X_{n'}, \Omega_n\}$. Then $\tilde{T}_i \cup T'$ is also an admissible set for each $i \geq k$ and

$$\begin{aligned} \varnothing \subset \cdots \subset \tilde{T}_{k-1} \subset \tilde{T}_k \subset \left(\tilde{T}_k \cup \{X_n, \Omega_n\}\right) \\ \subset \left(\tilde{T}_k \cup T'\right) \subset \cdots \subset \left(\tilde{T}_{2n-2} \cup T'\right) = \wp_n \end{aligned}$$

is a chain of length $2n$. ∎

THEOREM 3.3. *For any admissible set T,*

$$GK\left(\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle\right) = \operatorname{depth}(T) = 2n - \operatorname{ht}(T) = 2n - \operatorname{length}(T).$$

Proof. By 2.7 and [MR, 8.3.6(i)], we have $\operatorname{depth}(T) \leq GK(\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle) \leq 2n - \operatorname{ht}(T)$. Thus $GK(\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle) = \operatorname{depth}(T) = 2n - \operatorname{ht}(T)$ by 3.2.

It is easy to find a chain of admissible sets contained in T of length $\operatorname{length}(T)$. Thus we have $GK(\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle) \leq 2n - \operatorname{length}(T)$ by 2.7 and [MR, 8.3.6(i)].

Let $T = T_1 \cup \cdots \cup T_r$ be the connected decomposition. Put $k = \min(\operatorname{ind}(T_i))$, $m = \max(\operatorname{ind}(T_i))$. Define, for each $1 \leq i \leq r$,

$$A_i = \begin{cases} \{X_1, X_{1'}, \ldots, X_m, X_{m'}\} - T_i, & k = 1, \\ \{X_k, X_{k+1}, X_{(k+1)'}, \ldots, X_m, X_{m'}\} - T_i, & k > 1, \end{cases}$$

$$A_0 = \{X_j, \Omega_j \mid \Omega_j \notin T\},$$

$$A = A_0 \cup A_1 \cup \cdots \cup A_r,$$

and let R be the subalgebra generated by A. Then $GK(R) = |A|$ because for all $a, b \in A$, $ab = \lambda_{ab} ba$ for some $0 \neq \lambda_{ab} \in \mathbf{C}$. Since $|A| = 2n - \operatorname{length}(T)$ and $R \cap \langle T\rangle = 0$, we have $GK(\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle) \geq 2n - \operatorname{length}(T)$. This completes the proof. ∎

LEMMA 3.4. *Let T be an admissible set of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$. Then there exist $m \leq n$ and an admissible set T' of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2m})$ with no removable indices such that $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle \cong \mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2m})/\langle T'\rangle$.*

Proof. Let $i \in \text{ind}(T)$ be a removable index. Let $i > 2$. Define a map $\phi: \mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n}) \to \mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n-2})/\langle \Omega_{i-1}\rangle$ by

$$\phi(X_k) = \begin{cases} X_k & \text{if } 1 \le k < i, \\ 0 & \text{if } k = i \text{ or } k = i' \\ X_{k-1} & \text{if } i < k < i', \\ q^{-1}X_{k-2} & \text{if } i' < k \le 2n. \end{cases}$$

Clearly the ideal $\langle X_i, X_{i'}\rangle$ is contained in the kernel of ϕ and we have a chain of prime ideals $\langle 0\rangle \subset \langle \Omega_{i-1}\rangle \subset \langle X_i\rangle \subset \langle X_i, X_{i'}\rangle$ in $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$. Thus we have $GK(\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle X_i, X_{i'}\rangle) \le 2n-3$ by [MR, 8.3.6(i)]. Since the Gelfand–Kirillov dimension of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n-2})/\langle \Omega_{i-1}\rangle$ is $2n-3$ by 3.3, we have $\ker(\phi) = \langle X_i, X_{i'}\rangle$. Hence, $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle X_i, X_{i'}\rangle$ is isomorphic to $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n-2})/\langle \Omega_{i-1}\rangle$ via ϕ and

$$\langle \phi(T)\rangle \cap \{X_1, X_{1'}, \dots, X_{n-1}, X_{(n-1)'}, \Omega_1, \dots, \Omega_{n-1}\}$$

is admissible in $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n-2})$ containing Ω_{i-1} by 1.5. Moreover, $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle \cong \mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n-2})/\langle \phi(T)\rangle$.

Let $i = 2$. Then $X_1 \in T$ or $X_{1'} \in T$, say $X_{1'} \in T$. Repeat the argument of the case $i > 2$ by replacing Ω_{i-1} by $X_{1'}$. The case $i = 1$ is very easy.

The result then follows by induction on the number of removable indices. ∎

4. Quotient of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle$

By 2.7, the algebra $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle$ is a noetherian domain for each admissible set T. Thus there exists its left quotient $Q(\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle)$ which is a division algebra. In this section, we find a nontrivial central element of the form ab^{-1} in $Q(\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle)$, where T is a connected admissible set of odd length. Therefore we see that every primitive ideal containing $\langle T\rangle$ has an element of the form $a - \alpha b$ for some $\alpha \in \mathbf{C}$ from [Ro, Sect. 8.4]. Hence, in 4.2, we can find an ideal $P_T(\boldsymbol{\alpha})$ which is a candidate to be primitive ideal.

LEMMA 4.1. *Let T be a connected admissible set of odd length. Put $i = \min(\text{ind}(T))$, $j = \max(\text{ind}(T))$. Then there are monomials $a, b \in \mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ such that*

(1) *a, b are products of X_k's, $i \le k \le j+1$, $i' \ge k \ge (j+1)'$.*

(2) *ab^{-1} is a nontrivial central element of the left quotient $Q(\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle)$.*

Hence, if P is a prime ideal containing $\langle T\rangle$ such that $b \notin P$ then ab^{-1} is a central element of the left quotient $Q(\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/P)$.

Proof. We may assume that all elements of $\operatorname{ind}(T)$ are not removable by 3.4.

Case 1. $i = 1$, $j < n$: Notice that j is odd by 3.3. Put $\{X_{i_1}, \ldots, X_{i_{j+2}}\} = \{X_1, X_{1'}, \ldots, X_{j+1}, X_{(j+1)'}\} - T$ where $i_k < i_{k+1}$. Check from 1.1 that $(X_{i_1}^{\varepsilon_1} X_{i_3}^{\varepsilon_3} \cdots X_{i_{j+2}}^{\varepsilon_{j+2}})(X_{i_2}^{\varepsilon_2} X_{i_4}^{\varepsilon_4} \cdots X_{i_{j+1}}^{\varepsilon_{j+1}})^{-1}$ is central in $Q(\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle)$ where

$$\varepsilon_k = \begin{cases} 1 & \text{if } X_{i_k} \in \{X_{j+1}, X_{(j+1)'}\}, \\ 2 & \text{otherwise.} \end{cases}$$

Case 2. $1 < i$, $j = n$: Put $\{X_{i_1}, \ldots, X_{i_t}\} = \{X_i, X_{i'}, \ldots, X_n, X_{n'}\} - T$ where $i_k < i_{k+1}$. Notice that t is even by 3.3 and that $X_{i_1} = X_i$, $X_{i_t} = X_{i'}$. Check from 1.1 that $(X_{i_1}^{\varepsilon_1} X_{i_3}^{\varepsilon_3} \cdots X_{i_{t-1}}^{\varepsilon_{t-1}})(X_{i_2}^{\varepsilon_2} X_{i_4}^{\varphi_4} \cdots X_{i_t}^{\varepsilon_t})^{-1}$ is central in $Q(\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle)$ where

$$\varepsilon_k = \begin{cases} 1 & \text{if } X_{i_k} \in \{X_i, X_{i'}\}, \\ 2 & \text{otherwise.} \end{cases}$$

Case 3. $i = 1$, $j = n$: Notice that n is odd by 3.3. Put

$$\{X_{i_1}, \ldots, X_{i_n}\} = \{X_1, X_{1'}, \ldots, X_n, X_{n'}\} - T,$$

where $i_k < i_{k+1}$. Check from 1.1 that $(X_{i_1} X_{i_3} \cdots X_{i_n})(X_{i_2} X_{i_4} \cdots X_{i_{n-1}})^{-1}$ is central in $Q(\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle)$.

Case 4. $1 < i \le j < n$: Put $\{X_{i_1}, \ldots, X_{i_t}\} = \{X_i, X_{i'}, \ldots, X_{j+1}, X_{(j+1)'}\} - T$ where $i_k < i_{k+1}$. Notice that t is even by 3.3 and that $X_{i_1} = X_i$, $X_{i_t} = X_{i'}$. Check from 1.1 that $(X_{i_1}^{\varepsilon_1} X_{i_3}^{\varepsilon_3} \cdots X_{i_{t-1}}^{\varepsilon_{t-1}})(X_{i_2}^{\varepsilon_2} X_{i_4}^{\varepsilon_4} \cdots X_{i_t}^{\varepsilon_t})^{-1}$ is central in $Q(\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle)$ where

$$\varepsilon_k = \begin{cases} 1 & \text{if } X_{i_k} \in \{X_i, X_{i'}, X_{j+1}, X_{(j+1)'}\}, \\ 2 & \text{otherwise.} \end{cases} \quad ∎$$

4.2. Let T be an admissible set with a connected component T' of odd length. Then any primitive ideal $P \supseteq T$ contains an element of the form $a - \alpha b$ for some $\alpha \in \mathbf{C}$ for 4.1 and [Ro, Sect. 8.4], where a, b are monomials of 4.1. The element $a - \alpha b$ is a $\{\overline{X}_1, \overline{X}_{1'}, \ldots, \overline{X}_n, \overline{X}_{n'}\}$-normal element of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T'\rangle$. Thus it gives a motivation of the following definition.

DEFINITION. Let $T = T \cup \cdots \cup T_r$ be the connected decomposition of an admissible set T of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$.

(1) We define $\mathrm{cmp}(T)$ to be the number r of connected components and $\mathrm{ocmp}(T)$ to be the number of connected components of odd length.

(2) Define $\mathbb{T}^k = \mathbf{C}^* \times \cdots \times \mathbf{C}^*$, the k-dimensional torus.

(3) For each T_i of odd length, choose an $\alpha_i \in \mathbf{C}^*$. Define $Y_{T_i}(\alpha_i) = a - \alpha_i b$, $b <_o a$, where a, b are monomials found in 4.1 and the order relation "$<_o$" will be defined in 5.4. And then define, for each $i = 1, 2, \ldots, r$,

$$P_i = \begin{cases} \langle T_i \rangle & \text{if length}(T_i) \text{ is even} \\ \langle T_i, Y_{T_i}(\alpha_i) \rangle & \text{if length}(T_i) \text{ is odd,} \end{cases}$$

$$\boldsymbol{\alpha} = (\alpha_i) \in \mathbb{T}^k, \qquad k = \mathrm{ocmp}(T),$$

$$P_T(\boldsymbol{\alpha}) = \sum_{1 \le i \le r} P_i.$$

5. C-BASIS OF $\mathbb{C}_q(\mathfrak{sp}\mathbf{C}^{2n})/P_T(\boldsymbol{\alpha})$

The diamond lemma [B] is used to find a basis of polynomial algebras defined by some relations. It seems to be difficult to check whether the defining relations of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/P_T(\boldsymbol{\alpha})$ are reduction finite or not. In this section, we introduce a concept "monomially divisible" and start with an easy observation, Proposition 5.2, and use this to find a C-basis of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/P_T(\boldsymbol{\alpha})$. This result will be used to prove primeness of $P_T(\boldsymbol{\alpha})$ in the next section.

DEFINITION 5.1. Let S be a PBW-algebra with variables $Y_1, \ldots, Y_k$ over a subset R and let $a = Y_1^{t_1} \cdots Y_k^{t_k}$, $b = Y_1^{s_1} \cdots Y_k^{s_k}$ be two monomials of S. Then a is said to be *monomially divisible by* b (denoted by $b \| a$, $b \nparallel a$ if not) if $s_i \le t_i$ for each $i = 1, \ldots, k$.

PROPOSITION 5.2. *Let S be a PBW-algebra of type $I \subseteq (\mathbb{Z}_+)^k$ with variables $Y_1, \ldots, Y_k$ over* $\mathbf{C}$. *Thus $\mathfrak{U}_1 = \{\mathbf{Y}^{\mathbf{r}} = Y_1^{r_1}, \ldots, Y_k^{r_k} | \mathbf{r} \in I\}$ is a* $\mathbf{C}$-*basis of S. Let $f \in S$ be normal. For all $g \in S$ such that $gf \neq 0$, if gf has a monomial with nonzero coefficient which is monomially divisible by an element of $\{c_1, \ldots, c_m\} \subseteq \mathfrak{U}_1$, when it is expressed as a combination of elements of $\mathfrak{U}_1$, then*

$$\mathfrak{U}_2 = \{\overline{\mathbf{Y}^{\mathbf{r}}} | \mathbf{r} \in I, c_i \nparallel \mathbf{Y}^{\mathbf{r}}, i = 1, \ldots, m\}$$

is $\mathbf{C}$-*linearly independent in* $S/\langle f \rangle$.

Proof. Since f is normal, all elements of $\langle f \rangle$ are of the form gf. Thus $\mathfrak{U}_2$ is linearly independent by our hypothesis. ∎

EXAMPLE 5.3. (1) Given the polynomial ring $\mathbf{C}[Y]$ and $f \in \mathbf{C}[Y]$ such that $\deg(f) = m$, the maximal monomial of $0 \neq gf$ is monomially divisible by Y^m. Thus $\{\bar{1}, \bar{Y}, \ldots, \bar{Y}^{m-1}\}$ is $\mathbf{C}$-linearly independent in $\mathbf{C}[Y]/\langle f \rangle$.

(2) Let $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be a basis for the Lie algebra $\mathfrak{sl}_2(\mathbf{C})$ and let $S = \mathbf{C}[e, h, f]$ denote the enveloping algebra of $\mathfrak{sl}_2(\mathbf{C})$. (The reader is referred, for example, to [Hu] for definitions and basic results.) The Casimir element $\Omega = 4ef + h^2 - 2h$ is central in S and $\{e^r h^s f^t \mid r, s, t = 0, 1, \ldots\}$ is a $\mathbf{C}$-basis of S. Since

$$\begin{aligned} e^r h^s f^t \Omega &= e^r h^s (4ef + h^2 - 2h) f^t \\ &= \alpha e^{r+1} h^s f^{t+1} + \beta e^r h^{s+2} f^t + \gamma e^r h^{s+1} f^t \end{aligned}$$

for some nonzero $\alpha, \beta, \gamma \in \mathbf{C}$, $0 \neq g\Omega$ has a monomial which is monomially divisible by ef. Thus $\{\overline{e^r h^s f^t} \mid ef \nmid e^r h^s f^t, r, s, t = 0, 1, \ldots\}$ is $\mathbf{C}$-linearly independent in $S/\langle \Omega \rangle$. (In fact, it is a $\mathbf{C}$-basis).

DEFINITION 5.4. We define two orders $<_o$, $<_r$ on $\mathbb{Z}^{2n}$ as follows:

$$\begin{aligned} &(r_1, r_2, \ldots, r_{2n}) <_o (s_1, s_2, \ldots, s_{2n}), \\ &(r_1, r_2, \ldots, r_{2n}) <_r (s_1, s_2, \ldots, s_{2n}) \end{aligned}$$

if and only if

$$\begin{aligned} &(r_n, r_{n'}, r_{n-1}, r_{(n-1)'}, \ldots, r_1, r_{1'}) < (s_n, s_{n'}, s_{n-1}, s_{(n-1)'}, \ldots, s_1, s_{1'}), \\ &(r_{1'}, r_1, r_{2'}, r_2, \ldots, r_{n'}, r_n) < (s_{1'}, s_1, s_{2'}, s_2, \ldots, s_{n'}, s_n) \end{aligned}$$

in the lexicographic order, respectively. Define an injective function φ: $\mathscr{A} \to \mathbb{Z}^{2n}$ by $\varphi(X_1^{r_1} X_{1'}^{r_2} \cdots X_{2n}^{r_{2n}}) = (r_1, r_2, r_3, r_4, \ldots, r_{2n})$.

We define order relations $<_o$, $<_r$ on the $\mathbf{C}$-basis $\mathscr{A}$ of $\mathscr{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ (see 1.10(2)) to be that

$$\begin{aligned} a &= X_1^{r_1} X_2^{r_2} \cdots X_{1'}^{r_{1'}} <_o b = X_1^{s_1} X_2^{s_2} \cdots X_{1'}^{s_{1'}}, \\ a &= X_1^{r_1} X_2^{r_2} \cdots X_{1'}^{R_{1'}} <_r b = X_1^{s_1} X_2^{s_2} \cdots X_{1'}^{s_{1'}} \end{aligned}$$

if $\varphi(a) <_o \varphi(b)$ and if $\varphi(a) <_r \varphi(b)$, respectively. For a factor algebra R of $\mathscr{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ and two monomials $\bar{a}, \bar{b} \in R$, $a, b \in \mathscr{A}$, we write $\bar{a} \leq_o \bar{b}$, $\bar{a} \leq_r \bar{b}$ provided $a \leq_o b$, and $a \leq_r b$, respectively.

LEMMA 5.5. *Let $T = T_1 \cup \cdots \cup T_r$ be a connected decomposition in $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$. Set*

$$J_1(T) = \begin{cases} \{1\} \cup \{j \mid j = \max(\operatorname{ind}(T_i)) + 1 \text{ for some } 1 \le i < r\}, & \\ & \text{if } 1 \notin \operatorname{ind}(T), \\ \{j \mid j = \max(\operatorname{ind}(T_i)) + 1 \text{ for some } 1 \le i \le r\}, & \\ & \text{if } 1 \in \operatorname{ind}(T), \end{cases}$$

$$J_1'(T) = \begin{cases} \{j \mid j = \min(\operatorname{ind}(T_i)) \text{ for some } 1 \le i \le r\}, & \text{if } 1 \notin \operatorname{ind}(T), \\ \{j \mid j = \min(\operatorname{ind}(T_i)) \text{ for some } 1 < i \le r\}, & \text{if } 1 \in \operatorname{ind}(T), \end{cases}$$

$$m_1(T) = (T \cap \{X_1, X_{1'}, \ldots, X_n, X_{n'}\}) \cup \{X_i X_{i'} \mid i \in J_1(T)\},$$
$$m_1'(T) = (T \cap \{X_1, X_{1'}, \ldots, X_n, X_{n'}\}) \cup \{X_i X_{i'} \mid i \in J_1'(T)\}.$$

Then

$$\mathfrak{B}_1 = \{\bar{a} \mid a \in \mathscr{A}, b \nmid a \ \forall b \in m_1(T)\},$$
$$\mathfrak{C}_1 = \{\bar{a} \mid a \in \mathscr{A}, b \nmid a \ \forall b \in m_1'(T)\}$$

are $\mathbf{C}$*-bases of* $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle$.

Proof. We may assume that T does not contain any removable indices by 3.4. For showing that $\mathfrak{B}_1$ is a basis, we use induction on the number of connected components. Put $T' = T \cup T_{r+1} = T_1 \cup \cdots \cup T_r \cup T_{r+1}$, $k = \max(\operatorname{ind}(T_r))$, $i = \min(\operatorname{ind}(T_{r+1}))$, $j = \max(\operatorname{ind}(T_{r+1}))$. By the induction hypothesis, $\mathfrak{B}_1$ is a $\mathbf{C}$-basis of $R = \mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle$. First let us find a basis of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T \cup \{\Omega_i\}\rangle \cong R/\langle \overline{\Omega}_i\rangle$.

For each $g \in R$ such that $g\overline{\Omega}_i \ne 0$, let $\overline{X_1^{r_1} \cdots X_{2n}^{r_{2n}}}$ be the maximal monomial of g in $(\mathfrak{B}_1, <_r)$, where $<_r$ is induced from $(\mathscr{A}, <_r)$. For expressing $g\overline{\Omega}_i$ as a combination of elements of $\mathfrak{B}_1$, look at the following:

$$\begin{aligned} g\overline{\Omega}_i &= \alpha\overline{X_1^{r_1} \cdots X_n^{r_n}(q^{i-k+1}X_{k+1}X_{(k+1)'} + \cdots + X_i X_{i'})X_{n'}^{r_{n'}} \cdots X_{1'}^{r_{1'}}} + (*) \\ &= \beta\overline{X_1^{r_1} \cdots X_{k+1}^{r_{k+1}+1} \cdots X_n^{r_n} X_{n'}^{r_{n'}} \cdots X_{(k+1)'}^{r_{(k+1)'}+1} \cdots X_{1'}^{R_{1'}}} + \text{lower terms.} \end{aligned}$$

Therefore, $g\overline{\Omega}_i$ has a monomial of $\mathfrak{B}_1$, which is monomially divisible by $X_{k+1}X_{(k+1)'}$ and thus

$$\mathfrak{B}_1' = \{\bar{a} \mid a \in \mathscr{A}, b \nmid a \ \forall b \in \{X_{k+1}X_{(k+1)'}\} \cup m_1(T)\}$$

is a $\mathbf{C}$-basis of $S = \mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T \cup \{\Omega_i\}\rangle$ by 5.2. (It is easy to prove spanning!)

If $i < j$ then either $X_{i+1} \in T_{r+1}$ or $X_{(i+1)'} \in T_{r+1}$ by assumption, say $X_{i+1} \in T_{r+1}$. It is easy to see that all nonzero elements of the ideal of S generated by $\overline{X}_{i+1}$ have a monomial of $\mathfrak{B}'_1$, which is monomially divisible by X_{i+1}. Thus

$$\mathfrak{B}''_1 = \left\{\bar{a} \mid a \in \mathscr{A}, b \nmid a \ \forall b \in \left\{X_{k+1}X_{(k+1)'}, X_{i+1}\right\} \cup m_1(T)\right\}$$

is a $\mathbf{C}$-basis of $\mathscr{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T \cup \{\Omega_i, X_{i+1}\}\rangle$ by 5.2. Continue this technique to find a $\mathbf{C}$-basis of $\mathscr{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T \cup T_{r+1} = T'\rangle$.

Similarly, $\mathfrak{C}_1$ is also a basis. ∎

THEOREM 5.6. *Let $T = T_1 \cup \cdots \cup T_r$, $m_1(T)$ be as in* 5.5. *Set*

$$J(T) = \left\{i \mid 1 \le i \le r, \text{length}(T_i) \text{ is odd}\right\}.$$

Put $Y_{T_i}(\alpha_i) = a_i - \alpha_i b_i$, $b_i <_o a_i$ as in 4.2(3) *when $i \in J(T)$, and for $i \in J(T)$, set*

$$i_0 = \begin{cases} \max(\text{ind}(T_i)) + 1, & \text{if } \max(\text{ind}(T_i)) < n, \\ \max(\text{ind}(T_i)), & \text{if } \max(\text{ind}(T_i)) = n. \end{cases}$$

Define

$$m(T) = \begin{cases} m_1(T) \cup \left\{a_i, b_i X_{i'_0} \mid 1 \le i < r, i \in J(T)\right\} \cup \{a_r\}, & \text{if } r \in J(T) \\ m_1(T) \cup \left\{a_i, b_i X_{i'_0} \mid 1 \le i < r, i \in J(T)\right\}, & \text{if } r \notin J(T). \end{cases}$$

Then $\mathfrak{B} = \{\bar{a} \mid a \in \mathscr{A},\ b \nmid a\ \forall b \in m(T)\}$ is a $\mathbf{C}$-basis of $\mathscr{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/P_T(\boldsymbol{\alpha})$.

Proof. Put $R = \mathscr{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle$. Notice that $\mathscr{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle P_T(\boldsymbol{\alpha})\rangle$ is isomorphic to $R/\langle\{\overline{Y}_{T_i}(\alpha_i) \mid i \in J(T)\}\rangle$ and that $\overline{Y}_{T_i}(\alpha_i) = \bar{a}_i - \alpha_i \bar{b}_i$, $i \in J(T)$ is $\{\overline{X}_i \mid 1 \le i \le 2n\}$-normal in the factor algebra R by 4.1. Let us prove the result by using induction on $\{i_1 < i_2 < \cdots < i_t\} = J(T)$. Assume that $i = i_1 = \min(J(T)) < r$ and we will prove that

$$\mathscr{E} = \left\{\bar{a} \mid a \in \mathscr{A}, b \nmid a \ \forall b \in \left\{a_i, b_i X_{i'_0}\right\} \cup m_1(T)\right\}$$

is a $\mathbf{C}$-basis of $\mathscr{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T, Y_{T_i}(\alpha_i)\rangle$.

Set $j = \max(\text{ind}(T_i))$, $k = \min(\text{ind}(T_{i+1}))$. Notice that $i_0 = j + 1$, that a_i, b_i are monomially divisible by $X_{j+1}, X_{(j+1)'}$, respectively, by Case 4 of the proof of 4.1, and that $\overline{X}_{(j+1)'}$ is $\{\overline{X}_i \mid 1 \le i \le 2n\}$-normal in R by 1.3(2). If $0 \ne f \in R$ is in the ideal $\langle \overline{Y}_{T_i}(\alpha_i)\rangle$, then there is an element $0 \ne g \in R$ such that $f = \overline{Y}_{T_i}(\alpha_i) g$. Express f and g as combinations of elements of $\mathfrak{B}_1$ of 5.5.

Suppose that every monomial of g is monomially divisible by $X_{(j+1)'}$. Then g can be written as $g = \overline{X}^p_{(j+1)'}(g_0 + g_1\overline{X}_{(j+1)'} + \cdots + g_s\overline{X}^t_{(j+1)'})$. Notice that all monomials of g_i are not monomially divisible by X_{j+1} for each i, because $p > 0$. Look at the formula

$$\begin{aligned} f &= \overline{Y_{T_i}(\alpha_i)}g \\ &= \left(\overline{a_i X_{(j+1)'}} - \alpha_i \overline{b_i X_{(j+1)'}}\right)\overline{X}^{p-1}_{(j+1)'}\left(g_0 + g_1\overline{X}_{(j+1)'} + \cdots + g_t\overline{X}^t_{(j+1)'}\right) \end{aligned}$$

and notice that $\overline{b_i X_{(j+1)'} g_t X^{t+p-1}_{(j+1)'}}$ contains a monomial with maximal degree with respect to $X_{(j+1)'}$ in f. This monomial is monomially divisible by $b_i X_{(j+1)'}$ and so f contains a monomial which is monomially divisible by $b_i X_{(j+1)'}$.

Similarly, if there is a monomial of g which is not monomially divisible by $X_{(j+1)'}$ then g can be written as $g = g_0 + \overline{X}_{(j+1)}g_1 + \cdots + \overline{X}^t_{(j+1)}g_s$, where g_s does not contain any monomial which is monomially divisible by $X_{(j+1)'}$. Look at the formula

$$f = \overline{Y_{T_i}(\alpha_i)}g = \left(\bar{a}_i - \alpha_i \bar{b}_i\right)\left(g_0 + \overline{X}_{(j+1)}g_1 + \cdots + \overline{X}^s_{(j+1)}g_s\right).$$

and notice that $\overline{a_i X^s_{(j+1)} g_s}$ contains a monomial with maximal degree with respect to X_{j+1} in f. This monomial is monomially divisible by a_i and so f contains a monomial which is monomially divisible by a_i. Therefore $\mathscr{E}$ is linearly independent in $\mathscr{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T, Y_{T_i}(\alpha_i)\rangle$.

Since $\bar{a}_i = \alpha_i \bar{b}_i$,

$$\alpha_i \overline{b_i X_{(j+1)'}} = \overline{a_i X_{(j+1)'}} = \beta\left(\bar{a}_i/\overline{X}_{(j+1)}\right)\left(\overline{X_{j+1} X_{(j+1)'}}\right)$$

for some nonzero scalar β and $\mathfrak{B}_1$ spans $\mathscr{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T\rangle$, $\mathscr{E}$ spans the algebra $\mathscr{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T, Y_{T_i}(\alpha_i)\rangle$.

The case $i = i_1 = \min(J(T)) = r$ and the induction step are proved as in the case $i_1 < r$. They are left to the reader. ∎

LEMMA 5.7. *Let $T, T_i, m_1(T)$ be as in 5.5, let i_0 be as in 5.6, and let $T' = T_1 \cup \cdots \cup T_{r-1}$. Set*

$$m_2(T) = m_1(T) \cup \left\{a_i, b_i X_{i_0} \mid i \in J(T')\right\}.$$

If $T_r = \{\Omega_n\}$ then $\mathscr{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T \cup \{Y_{T_i}(\alpha_i) | i \in J(T')\}\rangle$ is a PBW-algebra of type $(\mathbb{Z}_+)^2$ with variables $X_n, X_{n'}$ over R, where R is all $\mathbf{C}$-combinations of elements of

$$\mathfrak{B}_3 = \left\{\overline{X_1^{s_1} \cdots X_{n-1}^{s_{n-1}} X_{(n-1)'}^{s_{(n-1)'}} \cdots X_{1'}^{s_{1'}}} \,\middle|\, b \nmid X_1^{s_1} \cdots X_{n-1}^{s_{n-1}} X_{(n-1)'}^{s_{(n-1)'}} \cdots X_{1'}^{s_{1'}} \quad \forall b \in m_2(T)\right\}.$$

Proof. The inductive step in the proof of 5.6 shows that

$$\mathfrak{B}_2 = \{\bar{a} \mid a \in \mathscr{A},\ b \nparallel a\ \forall b \in m_2(T)\}$$

is a $\mathbf{C}$-basis of $\mathscr{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T \cup \{Y_{T_i}(\alpha_i) \mid i \in J(T')\}\rangle$. Hence the result follows immediately. ∎

6. Complete Primeness of $P_T(\boldsymbol{\alpha})$

This section serves for proving complete primeness of $P_T(\boldsymbol{\alpha})$. The proof uses the Euclidean algorithm and the $\mathbf{C}$-basis of $\mathscr{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/P_T(\boldsymbol{\alpha})$ which is found in the previous section.

Proposition 6.1. *Let an integral domain S be a PBW-algebra of type $(\mathbb{Z}_+)^m$ over a subset R, namely, there are variables $Y_1, \dots, Y_m \in S$ such that all elements of S are uniquely expressed as a combination of monomials of Y_i's over R. Assume that*

$$\begin{aligned} Y_iY_j &= \lambda_{ij}Y_jY_i, & \lambda_{ij} &\in \mathbf{C}^* \\ a &= \beta Y_{i_1}^{s_{i_1}} \cdots Y_{i_p}^{s_{i_p}}, & \beta &\in \mathbf{C}^* \\ b &= \gamma Y_{j_1}^{s_{j_1}} \cdots Y_{j_r}^{s_{j_r}}, & \gamma &\in \mathbf{C}^* \\ \varnothing &= \{Y_{i_1}, \dots, Y_{i_p}\} \cap \{Y_{j_1}, \dots, Y_{j_r}\}. \end{aligned}$$

If $s_{i_k} = 1$ for some $1 \le k \le p$, a, b are both nontrivial monomials, and $a - b$ is a $\{Y_1, \dots, Y_m\}$-normal element of S, then the ideal $\langle a - b\rangle$ is completely prime in S.

Proof. For simplicity, set $s_{i_p} = 1$. The set

$$\mathfrak{F} = \left\{Y_{i_1}^{s_{i_1}} \cdots Y_{i_{p-1}}^{s_{i_{p-1}}} \mid s_{i_k} = 0, 1, \dots, 1 \le k \le p-1\right\}$$

is a left denominator set of S since $RY_i = Y_iR$. Hence there is the localization $S_{\mathfrak{F}}$.

If $f \in (a-b)S_{\mathfrak{F}}$ then there are $h, r \in S_{\mathfrak{F}}$ such that $f = h(a-b) + r$ and the degree of r with respect to Y_{i_p} is zero by the Euclidean algorithm.

Given $f, g \in \langle a - b\rangle \subseteq S$, write $f = f_1(a-b) + r_1$, $g = g_1(a-b) + r_2$ where $f_1, g_1, r_1, r_2 \in S_{\mathfrak{F}}$ and the degrees of r_1, r_2 with respect to Y_{i_p} are both zero. If $fg \in \langle a - b\rangle$ then we have $r_1r_2 \in S_{\mathfrak{F}}(a-b)$. Hence $r_1 = 0$ or $r_2 = 0$, say $r_1 = 0$, by comparing the degree with respect to Y_{i_p}. Thus

$f \in S_{\mathfrak{F}}(a-b)$ and there exist $Y_{i_1}^{t_{i_1}} \cdots Y_{i_{p-1}}^{t_{i_{p-1}}}$ and $e \in S$ such that $e = e_0 + \cdots + e_m$ is not left divisible by Y_{i_k} such that $t_{i_k} > 0$ and

$$Y_{i_1}^{t_{i_1}} \cdots Y_{i_{p-1}}^{t_{i_{p-1}}} f = e_0(a-b) + e_1(a-b) + \cdots + e_m(a-b). \quad (*)$$

Suppose $t_{i_k} > 0$ for some k and let e_j be not monomially divisible by Y_{i_k}. Then $e_j b$ is not monomially divisible by Y_{i_k} but all monomials of the left side of $(*)$ are monomially divisible by Y_{i_k}. Therefore we have $f \in \langle a-b \rangle$. ∎

THEOREM 6.2. *Let T be an admissible set of the algebra $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$. Then $P_T(\boldsymbol{\alpha})$, $\boldsymbol{\alpha} \in \mathbb{T}^k$, $k = \operatorname{ocmp}(T)$, is completely prime.*

Proof. We may assume that all $i \in \operatorname{ind}(T)$ are not removable by 3.4. Let $T = T_1 \cup \cdots \cup T_r$ be the connected decomposition. The proof is by induction on r. If $r = 0$ then $P_T(\boldsymbol{\alpha}) = \langle 0 \rangle$ is completely prime. Assume that the theorem is true for each admissible set such that the number of its connected components is less than r. Set $i = \min(\operatorname{ind}(T_r))$, $j = \max(\operatorname{ind}(T_r))$. let U be the subalgebra of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ generated by the elements

$$\begin{array}{ll} X_1, X_{1'}, \ldots, X_{j+1}, X_{(j+1)'}, & \text{if } j < n, \\ X_1, X_{1'}, \ldots, X_j, X_{j'}, & \text{if } j = n, \end{array}$$

and let P' be the ideal of U generated by $T_1 \cup \cdots \cup T_r \cup \{Y_{T_k}(\alpha_k) \mid k \in J(T)\}$. Then U is isomorphic to

$$\begin{array}{ll} \mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2j+2}) & \text{if } j < n, \\ \mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2j}) & \text{if } j = n. \end{array}$$

For $j < n$, since the ideal P' of U is σ_k-, δ_k-invariant for all $j+1 < k < (j+1)'$, where σ_k, δ_k are those of the proof of 1.10, $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/P_T(\boldsymbol{\alpha})$ is isomorphic to the iterated Ore extension $(U/P')[X_{j+2}, X_{(j+2)'}, \ldots, X_n, X_{n'}]$ by 2.5. Therefore it is enough to show that U/P' is an integral domain. Set

$$\{X_{i_1}, X_{i_2}, \ldots, X_{i_t}\} = \begin{cases} \{X_i, X_{i'}, \ldots, X_{j+1}, X_{(j+1)'}\} - T_r, & \text{if } j < n, \\ \{X_i, X_{i'}, \ldots, X_j, X_{j'}\} - T_r, & \text{if } j = n, \end{cases}$$

where $i_1 < i_2 < \cdots < i_t$.

If $i = 1$ then $T = T_r$ and $U/\langle T_r \rangle$ is isomorphic to the iterated Ore extension $S = \mathbf{C}[X_{i_1}, X_{i_2}, \ldots, X_{i_t}]$. If length$(T_r)$ is even then $P' = \langle T_r \rangle$ is completely prime in U. If length(T_r) is odd then $Y_{T_r}(\alpha_r)$ is of the form

$a - b$ of 6.1 in S by the proof of 4.1 and X_{i_k}'s satisfy the condition of 6.1; thus $P' = \langle T \cup \{Y_{T_r}(\alpha_r)\}\rangle$ is completely prime in U.

If $i > 1$, let V_1, V_2 be the subalgebras of U generated by $X_1, X_{1'}, \dots, X_{i-1}, X_{(i-1)'}$ and $X_1, X_{1'}, \dots, X_i, X_{i'}$, respectively, and let $T' = T_1 \cup T_2 \cdots \cup T_{r-1}$. Then V_1 is isomorphic to $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2i-2})$ and the factor algebras

$$W = V_1/\langle T' \cup \{Y_{T_k}(\alpha_k) | k \in J(T')\}\rangle$$

is an integral domain by the induction hypothesis and $V_2/\langle T' \cup \{Y_{T_k}(\alpha_k) | k \in J(T')\}\rangle$ is isomorphic to the iterated Ore extension $W[X_i, \sigma_i', \delta_i'; X_{i'}, \sigma_{i'}', \delta_{i'}']$ where $\sigma_i', \delta_i', \sigma_{i'}', \delta_{i'}'$ are induced from $\sigma_i, \delta_i, \sigma_{i'}, \delta_{i'}$ defined in the proof of 1.10. By 2.3, the ideal $\langle \overline{\Omega}_i \rangle$ of $W[X_i, X_{i'}]$ generated by the natural image of Ω_i is completely prime as in the proof of 2.6. The factor algebra

$$W[X_i, X_{i'}]/\langle \overline{\Omega}_i \rangle \cong V_2/\langle T' \cup \{\Omega_i\} \cup \{Y_{T_k}(\alpha_k) | k \in J(T')\}\rangle$$

is a PBW-algebra of type $(\mathbb{Z}_+)^2$ with variables $X_i = X_{i_1}, X_{i'} = X_{i_t}$ by 5.7. Notice that

$$U/\langle T \cup \{Y_{T_k}(\alpha_k) | k \in J(T')\}\rangle \cong \left(W[X_i, X_{i'}]/\langle \overline{\Omega}_i \rangle\right)\left[X_{i_2}, \dots, X_{i_{t-1}}\right].$$

Now the iterated Ore extension $(W[X_i, X_{i'}]/\langle \overline{\Omega}_i \rangle)[X_{i_2}, \dots, X_{i_{t-1}}]$ is a PBW-algebra of type $(\mathbb{Z}_+)^t$ with variables $X_{i_1}, X_{i_2}, \dots, X_{i_t}$ and thus if $r \in J(T)$, $\langle Y_{T_r}(\alpha_r)\rangle$ is completely prime in $(W[X_i, X_{i'}]/\langle \overline{\Omega}_i \rangle)[X_{i_2}, \dots, X_{i_{t-1}}]$ by 6.1. Therefore U/P' is an integral domain ∎

7. Primitive Ideals

In this section, we characterize all primitive ideals of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$. From the result, we see that its center is just $\mathbf{C}$ and the Gelfand–Kirillov dimensions of primitive factor algebras of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ are even. The difficulty is to prove the primitivity of the ideal $P_T(\boldsymbol{\alpha})$. We establish it by showing that the intersection of the primes strictly containing $P_T(\boldsymbol{\alpha})$ strictly contains $P_T(\boldsymbol{\alpha})$. (See [MR, 9.1.8] or [S2, 8.8].)

THEOREM 7.1. *Let*

$$\mathcal{P} = \{(T, \boldsymbol{\alpha}) | T \text{ is an admissible set}, \boldsymbol{\alpha} \in \mathbb{T}^k, k = \operatorname{ocmp}(T)\}.$$

Then the map χ: $(T, \boldsymbol{\alpha}) \mapsto P_T(\boldsymbol{\alpha})$ defines a bijection between $\mathcal{P}$ and the set Prim$(\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n}))$ of all primitive ideals of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$.

Proof. Let $T = T_1 \cup \cdots \cup T_r$ be the connected decomposition. We may assume that $\operatorname{ind}(T)$ does not contain any removable indices by 3.4. Put $k = \min(\operatorname{ind}(T_i))$, $m = \max(\operatorname{ind}(T_i))$. Define, for $1 \le i \le r$,

$$B_i = \begin{cases} \{X_1, X_{1'}, \ldots, X_m, X_{m'}\} - T_i, & i \notin J(T), k = 1, \\ \{X_k, X_{k+1}, X_{(k+1)'}, \ldots, X_m, X_{m'}\} - T_i, & i \notin J(T), k > 1, \\ \{X_1, X_{1'}, \ldots, X_m, X_{m'}, X_{m+1}\} - T_i, & i \in J(T), k = 1, m < n, \\ \{X_k, X_{k+1}, X_{(k+1)'}, \ldots, X_m, X_{m'}, X_{m+1}\} - T_i, & i \in J(T), k > 1, m < n, \\ \{X_k, X_{k+1}, X_{(k+1)'}, \ldots, X_{n-1}, X_{(n-1)'}\} - T_i, & i \in J(T), k > 1, m = n, \\ \{X_1, X_{1'}, \ldots, X_{n-1}, X_{(n-1)'}\} - T_i, i \in J(T), & k = 1, m = n, \end{cases}$$

and $B_0 = \{X_j, \Omega_j \mid \Omega_j \notin T, j \neq \max(\operatorname{ind}(T_i)) + 1 \text{ for some } i \in J(T)\}$. Let R be the subalgebra generated by $B = B_0 \cup B_1 \cup \cdots \cup B_r$. If $X_i \in B$ then $X_{i'} \notin B$. Thus for each $Y_i, Y_j \in B$, $Y_iY_j = q^{k_{ij}}Y_jY_i$, $k_{ij} \in \{1, -1, 2, -2\}$ by 1.1 and 1.3(1). Check that $GK(R) = 2n - \operatorname{length}(T) - \operatorname{ocmp}(T)$ is even.

Let us prove that $R \cap P_T(\boldsymbol{\alpha}) = 0$. Put $R_0 = \mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/\langle T \rangle$, $R_k = R_{k-1}/\langle \overline{Y}_{i_k} \rangle$, where $\{i_1 < \cdots < i_p\} = J(T)$. It is enough to show that the composition of the following natural maps

$$R \xrightarrow{\varphi_0} R_0 \xrightarrow{\varphi_1} R_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{p-1}} R_{p-1} \xrightarrow{\varphi_p} R_p \cong \mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/P_T(\boldsymbol{\alpha})$$

is injective. For each $0 \neq x \in R$, express $\varphi_0(x)$ as a combination of elements of $\mathfrak{C}_1$ of 5.5. Then we see immediately that $\varphi_0(x) \neq 0$. Next express $\varphi_0(x)$ as a combination of elements of $\mathfrak{B}_1$ and then the degree of $\varphi_0(x)$ with respect to $X_{u'}$ is zero for each $u \in J_1(T)$. Thus we have that $\varphi_k \cdots \varphi_1\varphi_0(x) \neq 0$ for each $k = 1, \ldots, p$. Notice that $GK(R) = GK(\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/P_T(\boldsymbol{\alpha}))$ because $GK(\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/P_T(\boldsymbol{\alpha})) \le 2n - \operatorname{length}(T) - \operatorname{ocmp}(T)$ and $R \cap P_T(\boldsymbol{\alpha}) = 0$.

It can be shown, by considering appropriate separate cases and using elementary row and column operations, that $\det(\log_q q^{k_{ij}}) = \det(k_{ij}) \neq 0$. Thus, by [MP, 1.3], every prime ideal $P \supsetneq P_T(\boldsymbol{\alpha})$ has a generator Y_i of R since the natural map $R \to \mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})/P$ is not injective and P is completely prime by 1.13. Hence $P_T(\boldsymbol{\alpha})$ is primitive by [MR, 9.1.8] or [S2, 8.8].

Let P be any primitive ideal of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$. Then $P \cap \wp = T$ is an admissible set by 1.5. Let T_i be a connected component of T such that $\operatorname{length}(T_i)$ is odd. Let a_i and b_i be a and b as in 4.2(3). Since a_i and b_i are products of certain X_t's which are normal modulo $\langle T_i \rangle$ and which are not contained in T, it is not possible for P to contain a_i or b_i. Thus P contains $P_T(\boldsymbol{\alpha})$ for some $\boldsymbol{\alpha}$ by 4.1 and [Ro, 8.4.18]. Since $P \cap \wp = T$, it follows from the paragraphs above that $P = P_T(\boldsymbol{\alpha})$. Hence the map χ is surjective.

Finally, if $P_T(\boldsymbol{\alpha}) = P_{T'}(\boldsymbol{\alpha}')$ then $T = P_T(\boldsymbol{\alpha}) \cap \wp = P_{T'}(\boldsymbol{\alpha}') \cap \wp = T'$ and so $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$ by 4.2. Hence χ is injective. ∎

COROLLARY 7.2. *The algebra $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ is primitive and its center is* $\mathbf{C}$.

Proof. The empty set is clearly admissible and the ideal $P_{\varnothing}(\boldsymbol{\alpha})$ is $\langle 0 \rangle$. Therefore $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ is primitive and its center is just $\mathbf{C}$. ∎

COROLLARY 7.3. *The Gelfand–Kirillov dimensions of primitive factor algebras of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^{2n})$ are even.*

APPENDIX

Admissible sets and their corresponding primitive ideals of $\mathcal{O}_q(\mathfrak{sp}\mathbf{C}^4)$:

$$\begin{aligned}
&\varnothing \leftrightarrow \langle 0 \rangle,\\
&\{\Omega_2\} \leftrightarrow \langle \Omega_2, X_2 - \alpha X_3 \rangle, \{X_1, \Omega_1\} \leftrightarrow \langle X_1, X_2 X_4^2 - \alpha X_3 \rangle,\\
&\{X_4, \Omega_1\} \leftrightarrow \langle X_4, X_1^2 X_3 - \alpha X_2 \rangle, \{X_1, X_4, \Omega_1\} \leftrightarrow \langle X_1, X_4 \rangle,\\
&\{X_1, X_2, \Omega_1, \Omega_2\} \leftrightarrow \langle X_1, X_2 \rangle, \{X_2, X_4, \Omega_1, \Omega_2\} \leftrightarrow \langle X_2, X_4 \rangle,\\
&\{X_1, X_3, \Omega_1, \Omega_2\} \leftrightarrow \langle X_1, X_3 \rangle, \{X_3, X_4, \Omega_1, \Omega_2\} \leftrightarrow \langle X_3, X_4 \rangle,\\
&\{X_1, X_2, X_3, \Omega_1, \Omega_2\} \leftrightarrow \langle X_1, X_2, X_3, X_4 - \alpha \rangle,\\
&\{X_1, X_2, X_4, \Omega_1, \Omega_2\} \leftrightarrow \langle X_1, X_2, X_3 - \alpha, X_4 \rangle,\\
&\{X_1, X_3, X_4, \Omega_1, \Omega_2\} \leftrightarrow \langle X_1, X_2 - \alpha, X_3, X_4 \rangle,\\
&\{X_2, X_3, X_4, \Omega_1, \Omega_2\} \leftrightarrow \langle X_1 - \alpha, X_2, X_3, X_4 \rangle,\\
&\{X_1, X_2, X_3, X_4, \Omega_1, \Omega_2\} \leftrightarrow \langle X_1, X_2, X_3, X_4 \rangle.
\end{aligned}$$

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